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THE MINIMUM WEIGHT DESIGN OF VIERENDEEL FRAMES

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Abstract—The present paper proposes a general analytical method of finding the linear minimum weight design of Vierendeel frames consisting of prismatic members of constant cross-section. The new concept of the "frame moment" which has been introduced by Tanabashi and Nakamura to the minimum weight design of tall multi-story multi-span building frames is shown to be useful and effective also for Vierendeel frames. Minimum weight designs are obtained for simply-supported and clamped Vierendeel frames, in simple and explicit forms. The principal advantage of the analytical solutions is that the general intrinsic features of the minimum weight designs can be revealed. A method of modifying the designs for the effect of axial forces is indicated.

1. INTRODUCTION

THE problem of minimum weight design of structural frames has been investigated by a number of authors since Foulkes [1] established the general theorems. Although the linear theory of minimum weight design of structural frames consisting of prismatic members of uniform cross-section has drawn considerable attention, previous investigations are mostly concerned with computational techniques [2–7], except Ref. [8]. While numerical methods are quite general, case by case numerical solutions would hardly clarify the general intrinsic features of the frames designed for minimum weight. A general solution even to a restricted class of frames, however, will reveal the intrinsic features of the class if obtained analytically and will also serve for practical purposes. From this point of view, the junior author has established in a simple and explicit analytical form the linear minimum weight design of a broad class of tall multi-story multi-span building frames subjected to large lateral forces [8]. In the present paper, a general method of finding the linear minimum weight designs is proposed for simply-supported and clamped Vierendeel frames based upon the concept of the "frame moment" [8].

A Vierendeel frame considered here consists of horizontal upper and lower chord members and vertical members as shown in Fig. 1. Each joint is assumed to be rigid enough to transmit the fully plastic moment of any member framing into it. All the assumptions

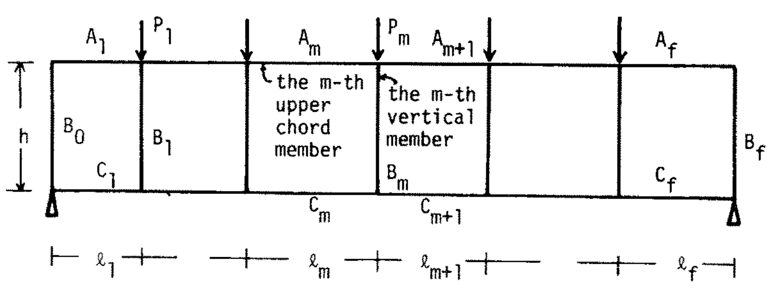

FIG. 1. Simply supported Vierendeel frame.

usually introduced in the linear theory of minimum weight design are also made here. The problem considered here may be stated as follows: Given the center line dimensions of a Vierendeel frame and a set of static loads to be carried as shown in Fig. 1, what fully plastic moments should be assigned to the members of uniform cross-sections in order to sustain the loads and to make the linear weight function a minimum? For the sake of simplicity, the paper considers the case where all the external loads act only upon the joints. Some local modification may be necessary if, at some subspans, the inter-span loads are not small compared to the subspan shear.

While a multi-span multi-story frame subjected to lateral loads may be regarded as a cantilever as a whole, a Vierendeel frame may be clamped and externally indeterminate. The weight function must therefore be minimized with respect to the unknown redundant reaction that governs the subspan shear.

2. EQUATIONS OF EQUILIBRIUM IN TERMS OF FRAME MOMENT

Consider first a simply-supported Vierendeel frame of f subspans or panels. Joints and vertical members are numbered from 0 through f from left, whereas chord members, from 1 through f as shown in Fig. 1. Let $l_1, l_2, \ldots$ and l_f denote the panel lengths, respectively and h, the height of the frame. Let A_j, C_j $(j = 1, 2, \ldots, f)$ and B_j $(j = 0, 1, 2, \ldots, f)$ denote the fully plastic moments of the jth upper and lower chord members and of the jth vertical member, respectively, from left.

Due to the assumption that all the external loads act upon the joints only, the end sections are the only potentially critical sections. In view of the overcomplete collapse mechanism for the minimum weight design of tall multi-story frames, it is natural to expect also that the minimum weight design of the Vierendeel frame be included in such a class of designs that correspond to extremely deteriorated overcomplete collapse mechanisms in which plastic hinges have formed at all the potentially critical sections as shown in Fig. 2. It should then be observed that, since all the end sections attain their fully plastic moments respectively, under such a special circumstance, the number of unknown end moments is equal to the number of members, provided that their signs are properly chosen. There are then $(2f+1)$ independent equations of joint equilibrium and f equations of sway equilibrium for the $(3f+1)$ unknown fully plastic moments and the problem may be said to be statically determinate. The essential problem is then reduced to how the directions of end moments should be determined.

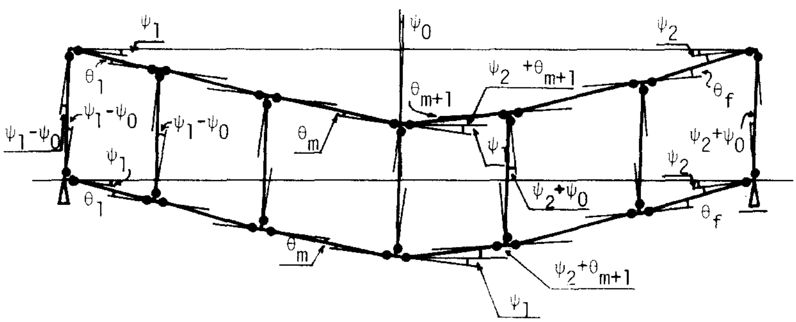

FIG. 2. Assumed overcomplete collapse mechanism.

The concept of "frame moment" is not only useful for a proper choice of end moment signs but also convenient for understanding the simple beam character of the Vierendeel frame. In view of the linearity of the equations of equilibrium in terms of plastic moments without any slope-continuity requirements, the moment diagram at collapse may be regarded as a composite consisting of the f constituent unit moment diagram shown in Fig. 3. A unit diagram is to be characterized by the particular moment distribution shown in Fig. 4 with the four corner values of equal magnitude. Such a unit moment diagram is completely

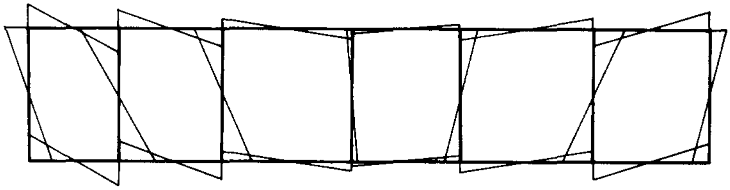

FIG. 3. Assumed moment diagram at collapse.

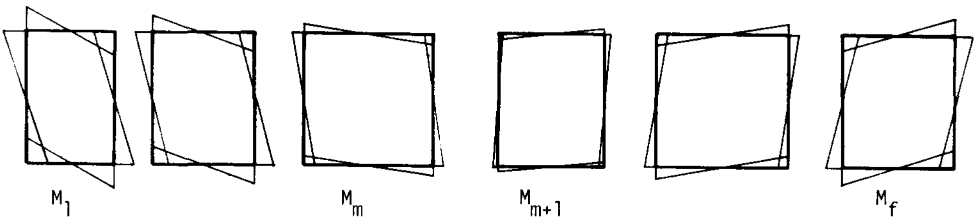

FIG. 4. Decomposed moment diagram for the definition of "frame moment".

defined by this equal corner moment except the direction of the corresponding shear force. The corner moment associated with this moment distribution is called "frame moment M_i" for the ith rectangle. The positive frame moment is defined here to be the one that corresponds to a clockwise couple of vertical shear forces Q_i in analogy to the usual shear force sign convention. Then Q_i is related to M_i by

$$4M_i = l_i Q_i \tag{1}$$

The fully plastic moments may be expressed in terms of M_i as follows:

$$\begin{aligned} A_j = C_j &= |M_j| \qquad (j = 1, 2, \ldots, f) \\ B_i &= |M_i + M_{i+1}| \qquad (i = 1, 2, \ldots, f-1) \\ B_0 &= |M_1|, \qquad B_f = |M_f|. \end{aligned} \tag{2}$$

In the present case of externally statically determinate frame, Q_i and hence M_i are immediately determined by the external loads. If the plastic moments are assigned to the members in accordance with equations (2), the moment distribution satisfies all the equations of equilibrium without violating any yield conditions and corresponds to the overcomplete collapse mechanism. Hence this is indeed a complete solution in the theory of limit analysis.

Because of the statical determinacy* in the sense stated above, there remains no further freedom for minimizing the linear weight function

$$G = 2g\,\Sigma\,|(l_i+h)M_i| = \tfrac{1}{2}g\,\Sigma\,|(l_i+h)l_iQ_i|, \tag{3}$$

(g: proportionality constant)

as far as the minimum weight solution is included in the assumed class of collapse mechanisms. Geometrically speaking, there is only one vertex determined by equations (1) and (2) in a $(3f+1)$-dimensional plastic moment space. In order to prove that this solution is indeed the minimum weight design, it is necessary to show that a Foulkes mechanism can be formed at this vertex. Otherwise, the minimum weight design has not been included in the assumed class of collapse mechanisms and another vertex must be sought at which the weight function can have a smaller value.

3. MECHANISM CONDITION

Since $(3f+1)$ members may be assigned different plastic moments in this problem, the Foulkes mechanism condition may be stated as follows [1]:

> A frame of minimum weight design must be able to collapse in an overcomplete collapse mechanism consisting of $(3f+1)$ independent alternative mechanisms, such that, for every member,
>
> Σ (Hinge rotations in a member) = a common constant $\times$ the length of the member.

When a Foulkes mechanism is to be constructed for a regular rectangular frame, it should first be observed that number of groups of members have the same lengths, respectively. All the vertical members of a Vierendeel frame have the same length h, whereas a pair of lower and upper chord members have the same length l_i. A pair of lower and upper chord members can have the same amount of hinge rotations, if the corresponding plastic hinge rotations are the same, respectively. A pair of the same hinge rotations can be produced not only by the vertical displacement of a pair of lower and upper joints but also by the rotations of such a pair of joints of the same magnitude and direction. A joint rotation causes a hinge rotation in a vertical member framing into the joint. Since all the vertical members have the same length, the plastic hinge rotations at the ends of every vertical member must be of the same magnitude, their directions being always consistent with the moment distribution determined above. With these observations, a Foulkes mechanism may be constructed as follows:

If the given vertical loads are relatively large compared to the horizontal shear resultant, there exist a pair of adjacent chord members between which the vertical shear changes the direction. Let m denote the index of the joint and the vertical member between such a pair of chord members. The moment distribution in this mth vertical member depends upon whether $|M_m| \gtreqless |M_{m+1}|$. Figure 2 shows the mechanism corresponding to the case where $|M_m| > |M_{m+1}|$. Let ψ_1 denote the angle of clockwise rotation of the $(m+1)$ joints from left

* The authors are grateful to a referee for calling their attention to the recent paper by Megarefs and Sidhu [10] which was published at about the same time as the present manuscript was submitted. The referee has pointed out that this statical determinacy is similar to the concept of "optimally determinate design" in [10]. It should be noted that the present paper will be concerned with a problem of spatially discrete variables of minimizing the absolute area of a subspan shear diagram with respect to only one of the two redundant reactions, whereas Magarefs and Sidhu have dealt with the problem of minimizing the absolute area of the moment diagram for a frame with continuously varying cross-section.

and ψ_2, the angle of counterclockwise rotation of the remaining $f-(m+1)$ joints. Let ψ_0 denote the clockwise angle of horizontal sway of vertical members. Then the Foulkes condition can be satisfied through the $(f+1)$ vertical members if

$$2(\psi_1-\psi_0) = h\Theta, \qquad 2(\psi_2+\psi_0) = h\Theta. \tag{4}$$

where Θ is a positive proportionality constant. Let $\theta_j(j = 1,\ldots,m)$ and $\theta_k(k = m+1,\ldots,f)$ denote the angles of hinge rotation due to the vertical displacements of joints whose directions are to be consistent with the moment distribution. The Foulkes condition requires

$$2\theta_j = l_j\Theta \tag{5}$$

$$2\theta_{m+1}+\psi_1+\psi_2 = l_{m+1}\Theta \tag{6}$$

$$2\theta_k = l_k\Theta. \tag{7}$$

The equation of compatibility of the collapse mechanism:

$$\sum_{j=1}^{m} (\theta_j+\psi_1)l_j - \sum_{k=m+1}^{f} (\theta_k+\psi_2)l_k = 0 \tag{8}$$

must further be required in order to satisfy the condition of the external supports. Equation (8) is new and was not necessary in Ref. [8] because the overall external supports of a multistory frame may be regarded as a cantilever. The $(f+3)$ unknown angles ψ_0, ψ_1, ψ_2, θ_j and θ_k can be determined by the $(f+3)$ equations of (4), (5), (6), (7) and (8). If the sum of the two equations in (4) is substituted into equation (6), θ_{m+1} is given by

$$\theta_{m+1} = \tfrac{1}{2}(l_{m+1}-h)\Theta. \tag{9}$$

Since all the hinge rotations must be non-negative in order to be consistent with the moment distribution, equation (9) requires

$$l_{m+1} \geq h. \tag{10}$$

It may now be concluded that the Foulkes condition is indeed satisfied by the assumed mechanism shown in Fig. 2 for those frames where equation (10) is satisfied and hence that the minimum weight design is indeed given by equations (1) and (2).

If the vertical shear does not change its direction due to a relatively large lateral shear, then equation (6) will not be necessary and hence a simpler Foulkes mechanism may readily be constructed.

For frames with $h > l_{m+1}$, it is apparent that the assumed angles of plastic hinge rotation exceed $l_{m+1}\Theta$ even if $\theta_{m+1} = 0$. In order to decrease the sum of the hinge rotation, it seems unavoidable that the angle of rotation of the mth joint be decreased from ψ_1. This change, however, would then result in violation of the Foulkes condition which had been satisfied for the mth vertical member. The latter difficulty may be resolved if the mth vertical member is assigned $B_m = 0$ so that any arbitrary amount of plastic hinge rotation within the member becomes permissible. In order to maintain the equilibrium of moments about the mth joint after this change, it is necessary that the left end moment of the $(m+1)$th chord members be equal to $|M_m|$. Since the vertical shear remains the same, the moment diagram for the $(m+1)$th chord members must have the same slope as before. The right end moments must then be equal to $|M_{m+1}|-\{|M_m|-|M_{m+1}|\} = 2|M_{m+1}|-|M_m| < |M_m|$ as shown in

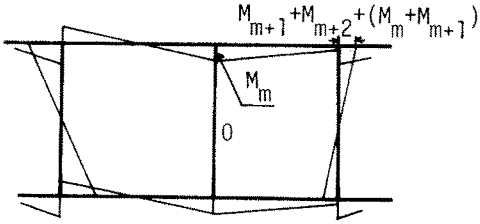

FIG. 5. Modified moment diagram at collapse.

Fig. 5. The weight change may be written as

$$2\{|M_m|-|M_{m+1}|\}(l_{m+1}-h) < 0.$$

Thus the weight is indeed decreased. Let the angle of clockwise rotation of the mth joint be ψ_m. The Foulkes condition will remain satisfied if

$$(\psi_1+\theta_m-\psi_m)+\theta_m = l_m\Theta \tag{11}$$

$$\psi_2+\psi_m = l_{m+1}\Theta. \tag{12}$$

If the sum of the two equations in (4) is subtracted from the sum of equations (11) and (12), then

$$2\theta_m = (l_m+l_{m-1}-h)\Theta.$$

The range of validity of this modified design is therefore given by

$$l_m+l_{m+1} \geq h. \tag{13}$$

It should be remarked that the mth vertical member in this modified design has lost the role as a bending-resistant member and simply behaves as a strut which transmits half the vertical load to the lower chord members.

4. CLAMPED VIERENDEEL FRAMES

A clamped Vierendeel frame shown in Fig. 6 is characterized by its external indeterminacy together with its end constraints against lateral sway. The circumstance that the vertical shear in each subspan is not statically determinate and dependent upon the redundant reaction, say R_1, in Fig. 6, indicates that the weight can be minimized with respect to R_1. Because of the end constraint against sway, on the other hand, the Foulkes mechanism in Fig. 2 must be so modified as to exclude the angle ψ_0.

The concept of the frame moment is again helpful for the minimization process with respect to the redundant reaction and enables one to regard the Vierendeel frame as a beam. Any moment distribution represented by a set of frame moments satisfying (1) provides a complete solution since it satisfies all the equilibrium equations, does not violate any yield conditions and corresponds to an overcomplete collapse mechanism. The corresponding design may readily be obtained from the frame moments as given by (2). Let D denote the set of all the designs that can be described by frame moments only. Assuming first that the minimum weight design $D_{\min}$ for a frame is included in D, the linear weight is minimized

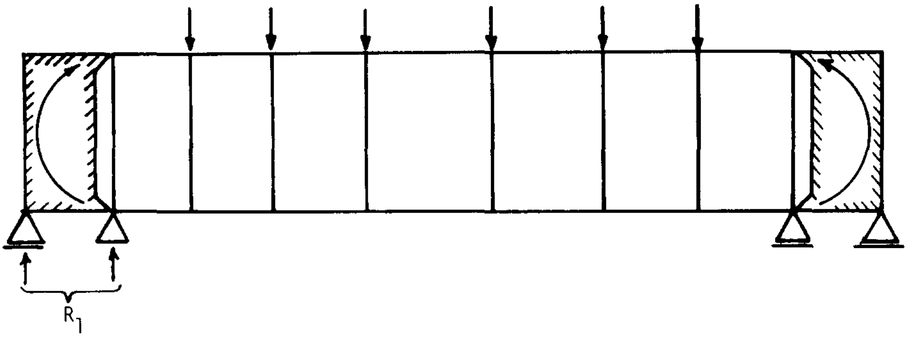

FIG. 6. Clamped Vierendeel frame.

with respect to the redundant reaction and the necessary conditions are found. In order to prove that D_{min} is indeed the minimum weight design among all the designs including those which can not be described by the frame moments only, it must be shown that the overcomplete collapse mechanism corresponding to D_{min} satisfies the Foulkes condition. It may then be shown that the necessary conditions defining essentially the region of applicability of D_{min} are equivalent to the conditions of nonnegative hinge rotations in the Foulkes mechanism. In this way the necessary and sufficient conditions for the minimum weight are found. There are frames, however, for which these conditions are violated only locally. This indicates that the minimal solution for such a frame has not been included in D. Because of the local violation of the Foulkes condition, however, it is natural to expect that a local modification on D_{min} obtained as above will suffice to achieve a different minimal solution. It will be shown through three successive modifications that a local modification on the moment diagram and the corresponding hinge pattern of the D_{min} indeed leads to a different minimal solution whose region of applicability is mutually exclusive and compensating with the previous design.

Design-I

For most practical purposes, it may be assumed that the vertical loads act only downward so that the corresponding transverse shear diagram is monotonically non-increasing with the shape as illustrated in Fig. 7. The weight function given by equation (3) indicates that, for a special case of equal subspan $l_i = l$, G is proportional to the area of the $|Q_i|$ diagram. It may readily be shown that if $\int_0^l |Q(x)|\,dx$ (where $Q(0) - Q(x)$ is a given continuous

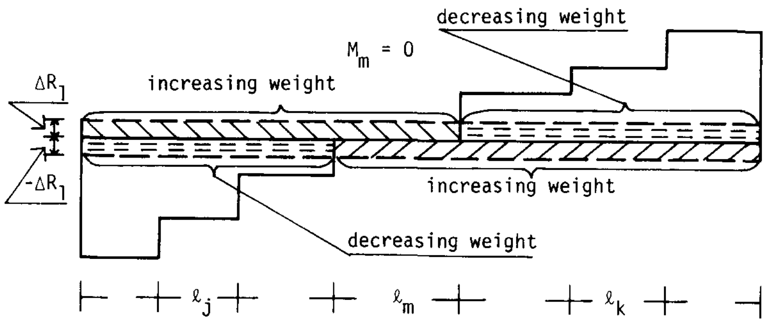

FIG. 7. Variations of the weight with respect to the redundant reaction R_1.

monotonic function) is to be minimized with respect to the end value $Q(0)$, then $Q(l/2) = 0$. Although the Q_i-diagram in the present problem is of a staircase shape, it is natural to expect that a minimizing Q_i-diagram would analogously contain a subspan with $Q_i = 0$ as illustrated in Fig. 7. Let n be an assumed index of the subspan for which $Q_n = 0$. Then the weight may be written as

$$G = \frac{1}{2}g\left\{\sum_{j=1}^{n-1}(l_j+h)l_jQ_j - \sum_{k=n+1}^{f}(l_k+h)l_kQ_k\right\}. \tag{14}$$

In view of Fig. 7 and the assumed monotonous variation of Q_i-diagram, it is apparent that G tends to decrease as n is increased from 1 in equation (14) and also decrease as n is decreased from f. There must therefore be a minimizing index number m between 1 and f.

Apparently, any variation of the reaction R_1 corresponds to a simple downward or upward translation of the Q_i-diagram. An increase ΔR_1 increases Q_j $(j = 1, \ldots, m-1)$ by ΔR_1, respectively and decreases $|Q_k|(k = m+1, \ldots, f)$ by ΔR_1, respectively. The corresponding variation in the weight may be calculated from the variations of the moment diagram as indicated by the dashed lines in Fig. 8(a). It should be noted that since $Q_m = 0$, the moment is zero through the mth chord members. Let ΔG_1 and ΔG_2 denote the weight variations for $\Delta R_1 > 0$ and for $-\Delta R_1$, respectively. In view of the circumstance that, since

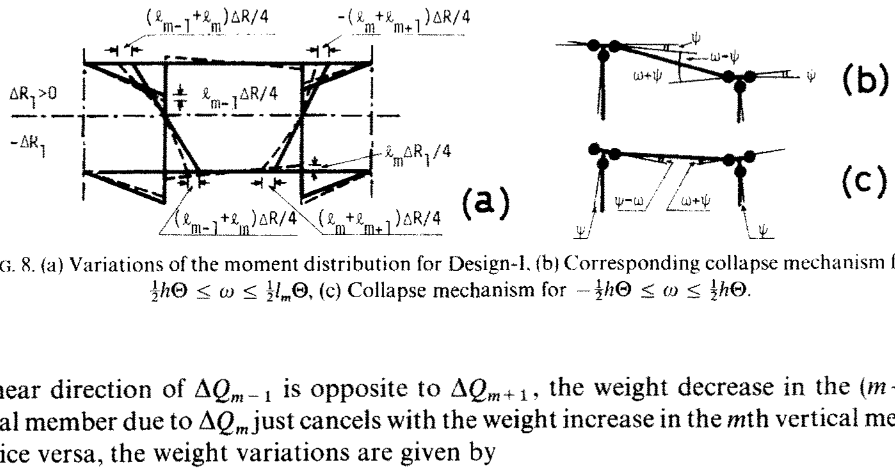

FIG. 8. (a) Variations of the moment distribution for Design-I, (b) Corresponding collapse mechanism for $\frac{1}{2}h\Theta \leq \omega \leq \frac{1}{2}l_m\Theta$, (c) Collapse mechanism for $-\frac{1}{2}h\Theta \leq \omega \leq \frac{1}{2}h\Theta$.

the shear direction of ΔQ_{m-1} is opposite to ΔQ_{m+1}, the weight decrease in the $(m+1)$th vertical member due to ΔQ_m just cancels with the weight increase in the mth vertical member and vice versa, the weight variations are given by

$$\Delta G_1 = \frac{1}{2}g\Delta R_1\left\{\sum_{j=1}^{m-1}(l_j+h)l_j + l_m^2 - \sum_{k=m+1}^{f}(l_k+h)l_k\right\}$$

$$\Delta G_2 = \frac{1}{2}g\Delta R_1\left\{-\sum_{j=1}^{m-1}(l_j+h)l_j + l_m^2 + \sum_{k=m+1}^{f}(l_k+h)l_k\right\}.$$

If the moment diagram shown in Fig. 8(a) is to correspond to the minimum weight design for a given frame, then $\Delta G_1 \geq 0$ and $\Delta G_2 \geq 0$. The necessary condition may be compactly written as

$$-l_m^2 \leq L(m) \leq l_m^2 \tag{15}$$

where

$$L(m) = \sum_{k=m+1}^{f} (l_k+h)l_k - \sum_{j=1}^{m-1} (l_j+h)l_j. \tag{16}$$

$L(m)$ is a discrete-valued function of the subspan index m and is characterized by the center line dimensions of a frame. It is therefore appropriate to call it the "shape function" for the frame. Although this function has been introduced here for the convenience of describing the weight variations, it will be shown later that $L(m)$ is closely related to the angle of rotation of the mth chord members in the corresponding collapse mechanism.

The moment diagram may also be varied under the same subspan shears. The only possible variation for the mth chord members is the constant moment distribution pattern similar to Fig. 9(a). This variation results in the necessary condition:

$$h \leq l_m. \tag{17}$$

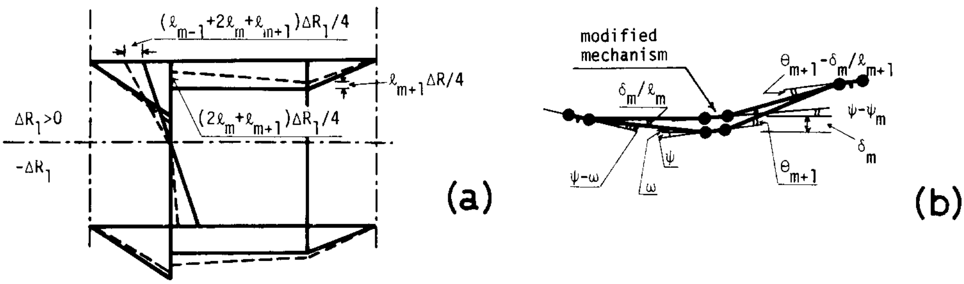

FIG. 9. (a) Variations of the moment distribution for Design-II, (b) Corresponding modified collapse mechanism.

For those frames for which (15) and (17) are satisfied, the minimizing Q_i-diagram belonging to the index m immediately determines the design through (1) and (2). In particular, $A_m = C_m = 0$ in this case. This design will be called Design-I.

In order to prove the sufficiency of (15) and (17), it must be shown that the Foulkes condition can be satisfied in the corresponding collapse mechanism. Because of the lateral constraint, $\psi_0 = 0$ and equation (4) are reduced to

$$\psi_1 = \psi_2 = \psi = \tfrac{1}{2}h\Theta. \tag{18}$$

The conditions (5) and (7) must remain satisfied except at the mth chord members. Figures 8(b) and (c) show two of the three possible patterns of hinge formation for Design-I. The angle of clockwise rotation of the mth chord members is obtained from the equation of compatibility for the mechanism as

$$\omega = \tfrac{1}{2}L(m)\Theta/l_m. \tag{19}$$

The conditions of non-negative hinge rotations in the mechanism shown in Fig. 8(c) are $\psi-\omega \geq 0$ and $\psi+\omega \geq 0$, from which the following inequalities are obtained.

$$-\tfrac{1}{2}h\Theta \leq \omega \leq \tfrac{1}{2}h\Theta. \tag{20}$$

Similarly, the conditions $\omega+\psi \leq 0$ and $\omega-\psi \geq 0$ for the mechanism in Fig. 8(b) lead to

$$\tfrac{1}{2}h\Theta \leq \omega \tag{21}$$

whereas the third case similar to Fig. 8(b) but just reversed, leads to

$$\omega \leq -\tfrac{1}{2}h\Theta. \tag{22}$$

Since the particular circumstance of $A_m = C_m = 0$ permits any increase in the amount of hinge rotations by assuming arbitrary plastic hinges within the chord members, the Foulkes condition can be satisfied if the total sum Ω_m of hinge rotations for an mth chord member is not greater than $l_m\Theta$. Hence

$$\Omega_m = 2|\omega| \leq l_m\Theta, \qquad \text{for Fig. 8(b) and}$$

$$\Omega_m = 2\psi \leq l_m\Theta \qquad \text{for Fig. 8(c).}$$

Thus the Foulkes condition can be satisfied if h and ω are within the rectangular region defined by $-\frac{1}{2}l_m\Theta \leq \omega \leq \frac{1}{2}l_m\Theta$ and $h \leq l_m$ on an (h, ω)-plane, which consists of the three subregions defined by (20), (21) and (22). These conditions obtained by the kinematical consideration precisely coincide with (15) and (17). It may therefore be concluded that the existence of the subspan index m satisfying (15) and (17) for a given set of centerline dimensions of a frame is the necessary and sufficient condition for the Design-I belonging to m to be the minimum weight design. In those frames for which $h > l_m$, Ω_m is in excess of $l_m\Theta$ and a different design must be sought.

Design-II

The state of $Q_m = 0$ does not necessarily imply $A_m = C_m = 0$ if designs other than in D may be considered. Figure 9(a) shows a constant moment distribution through the mth chord members, corresponding to the magnitude of which the plastic moments of the $(m-1)$th and the mth vertical members are decreased. This modification may be regarded as a process of shifting $\min\{B_{m-1}, B_m\}$ to the mth chord members due to the circumstance $h > l_m$. Figure 9(*a*) shows the case where $\min\{B_{m-1}, B_m\} = B_m$. If the modified values of the plastic moments are indicated by a bar, the modified design may be written as

$$\bar{A}_m = \bar{C}_m = B_m, \qquad \bar{B}_{m-1} = B_{m-1} - B_m, \qquad \bar{B}_m = 0. \tag{23}$$

The amount of weight decrease compared to Design-I is given by $2(h-l_m)\min\{B_{m-1}, B_m\}$. This design will be called Design-II. If the moment diagram shown in Fig. 9(a) is to correspond to the minimum weight design, any change of redundant reaction R_1 must increase the weight. The smallest possible weight variations as R_1 is changed by ΔR_1 are illustrated by the dashed lines in Fig. 9(a) for $\Delta R_1 > 0$ and $-\Delta R_1$ and may be expressed as

$$\Delta G_1 = \tfrac{1}{2}g\Delta R_1\{-L(m)+hl_m+(h-l_m)l_{m+1}\}$$

$$\Delta G_2 = \tfrac{1}{2}g\Delta R_1\{L(m)-(h-l_m)(l_m+l_{m+1})+l_m^2\}.$$

The necessary condition for the minimum weight is therefore given by

$$(h-l_m)(l_m+l_{m+1})-l_m^2 \leq L(m) \leq hl_m+(h-l_m)l_{m+1}. \tag{24}$$

Variations of the moment diagram without any change in the Q_i-diagram may again be considered. The variations with respect to a simple downward translation of the moment diagram of the mth and $(m+1)$th chord members and to an upward translation of the diagram of the mth chord members lead to the following inequalities

$$l_m \leq h \leq l_m + l_{m+1}. \tag{25}$$

Since the necessary conditions have been obtained with respect to the restricted class of variations, the sufficiency must be proved by constructing a Foulkes mechanism.

In view of the moment diagram shown in Fig. 9(a), the mechanism similar to the case Fig. 8(c) is the only possible one. Since $\Omega_m = 2\psi = h\Theta > l_m\Theta, \Omega_m$ must be decreased. Ω_m can be decreased if the angles of rotation of the mth joints are decreased by ψ_m and simultaneously an additional upward displacement δ_m of the mth joints are considered as shown in Fig. 9(b). This modification is permissible since the apparent decrease of the plastic hinge rotation in the mth vertical member with $\bar{B}_m = 0$ can be compensated by considering an arbitrary plastic hinge within the member. It may then be readily shown that the Foulkes condition can be satisfied if

$$\psi_m = (h - l_m)\Theta \geq 0 \quad \text{and} \quad \delta_m = \tfrac{1}{2}(h - l_m)l_{m+1}\Theta \geq 0. \tag{26}$$

Since all the hinge rotations must be non-negative, the following inequalities must be satisfied in view of Fig. 9(b).

$$\psi - \omega + \delta_m/l_m \geq 0, \quad \psi - \psi_m + \omega - \delta_m/l_m \geq 0, \quad \theta_{m+1} - \delta_m/l_{m+1} \geq 0, \quad \theta_{m+1} - \delta_m/l_{m+1} + \psi_m \geq 0 \tag{27a–d}$$

(27a,b) and (27c,d) together with (26) may be shown to be reduced to (24) and (25), respectively. The existence of the subspan index m satisfying (24) and (25) for a given frame is the necessary and sufficient condition for the Design-II belonging to m to be the minimum weight design.

Design-III

A different minimizing shear diagram must be sought for a frame for which there does not exist the index m satisfying (15) and (17) or (24) and (25). It should be pointed out here that both Design-I and Design-II have included a member whose plastic moment is zero. The plastic moment of the mth vertical member may be zero not only in the case of Design-II but also if $M_m + M_{m+1} = 0$ and hence if $Q_m l_m + Q_{m+1} l_{m+1} = 0$. The corresponding shear force diagram to this case crosses the zero line at the position of the joint m. This condition determines the shear force diagram and the frame moments uniquely. Variations of the moment diagram depend upon whether $h \gtreqless l_m$ and $h \gtreqless l_{m+1}$ as indicated in Fig. 10(a) and (b) by dashed lines for $\Delta R_1 > 0$ and for $-\Delta R_1$, respectively. If this moment diagram is to render the minimum weight, the necessary conditions may be written as

$$\begin{aligned}
L(m) &\geq l_m^2 && \text{if } h \leq l_m \\
L(m) &\geq h(l_m + l_{m+1}) - l_m l_{m+1} && \text{if } h \geq l_m \\
L(m) &\leq h(l_m + l_{m+1}) + l_m^2 && \text{if } h \leq l_{m+1} \\
L(m) &\leq l_{m+1}(l_m + l_{m+1}) + l_m^2 && \text{if } h \geq l_{m+1}.
\end{aligned} \tag{28a–d}$$

The collapse mechanism corresponding to the moment diagrams shown in Fig. 10(a) and (b) must be similar to that shown in Fig. 8(b). Since $\Omega_m = 2\omega = L(m)\Theta/l_m \geq l_m\theta$, from (28a,b), however, Ω_m must be decreased. With this motivation, the mechanism shown in Fig.

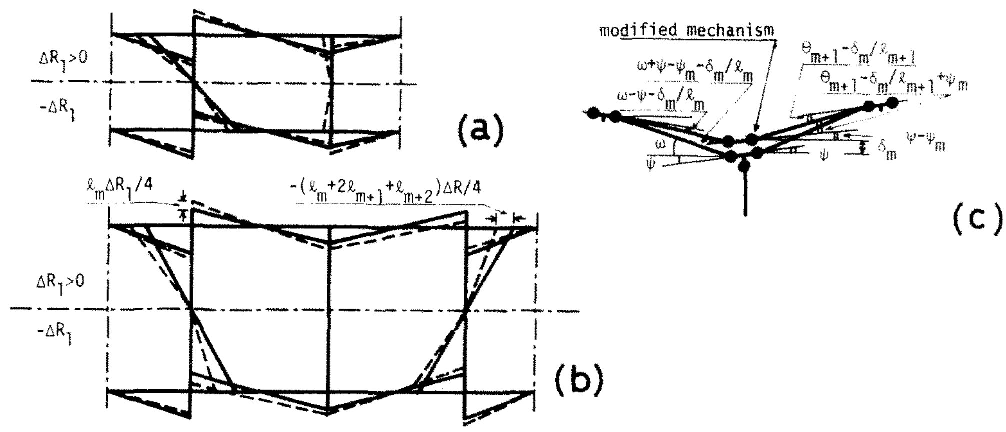

FIG. 10. (a), (b) Variations of the moment distribution for Design-III, (c) Corresponding modified collapse mechanism.

10(c) may be considered in which the angle of rotation of the mth joints has been decreased by ψ_m and simultaneously an additional upward displacement δ_m of the mth joint has been considered. These quantities may be so determined that the Foulkes condition is satisfied.

$$\delta_m = \frac{l_m l_{m+1}(\omega - \frac{1}{2}l_m\Theta)}{l_m + l_{m+1}} \geq 0, \qquad \psi_m = \frac{2l_m(\omega - \frac{1}{2}l_m\Theta)}{l_m + l_{m+1}} \geq 0. \tag{29}$$

It may readily be confirmed that the conditions of non-negative rotations on the four plastic hinges shown in Fig. 10(c) and $\psi - \psi_m \geq -\frac{1}{2}h\Theta$ are equivalent to the four inequalities (28a–d). The existence of the index number m satisfying (28a–d) is therefore the necessary and sufficient condition for Design-III to be the minimum weight design.

Design-IV

Design-II and Design-III have included respectively one vertical member whose plastic moment is zero. It is natural then to look for a design such that the plastic moments of two vertical members, say the $(m-1)$th and mth, are zero and such that, except in the mth sub-span, the moment distribution is described by the frame moments. This condition determines the Q_i-diagram and the corresponding moment diagram uniquely as shown in Fig. 11(a).

$$Q_m = \frac{l_{m+1}P_m - l_{m-1}P_{m-1}}{l_{m-1} + 2l_m + l_{m+1}}, \qquad Q_{m-1} = Q_m + P_{m-1}, \qquad Q_{m+1} = Q_m - P_m, \text{etc} \tag{30}$$

The dashed lines in Fig. 11(a) show two of the four possible variations of the moment diagram resulting in the smallest weight changes, one for $\Delta R_1 > 0$ and the other $-\Delta R_1$. It may be shown that, if this moment diagram is to correspond to the minimum weight design for a frame, the shape function must be within the following region.

$$\begin{aligned}
&L(m) \leq (h - l_m)(l_m + l_{m+1}) - l_m^2 && \text{if } l_m + l_{m+1} \geq h \\
&L(m) \leq l_{m+1}(l_{m-1} + 2l_m + l_{m+1}) + l_m l_{m-1} - h(l_{m-1} + l_m) && \text{if } l_m + l_{m+1} \leq h \\
&L(m) \geq -h(l_{m-1} + l_m) + l_{m-1}l_m && \text{if } l_{m-1} + l_m \geq h \\
&L(m) \geq (h - l_m)(l_m + l_{m+1}) - l_m^2 - l_{m-1}(l_{m-1} + 2l_m + l_{m+1}) && \text{if } l_{m-1} + l_m \leq h.
\end{aligned} \tag{31a–d}$$

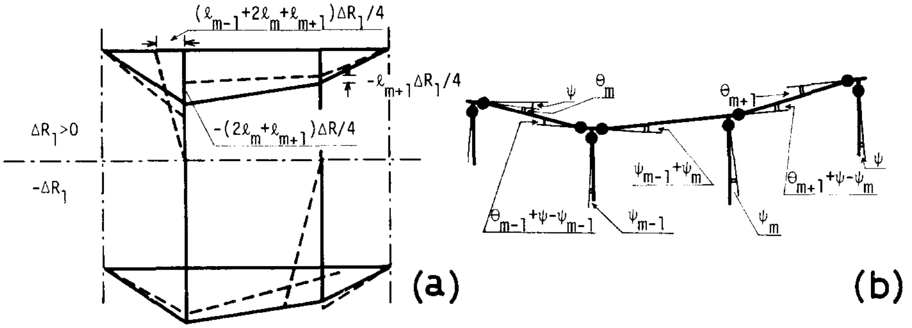

FIG. 11. (a) Variations of the moment distribution for Design-IV, (b) Corresponding collapse mechanism.

Figure 11(b) shows the corresponding Foulkes mechanism in which

$$\psi_m = \frac{-L(m)+l_{m-1}l_m+h(l_{m+1}-l_{m-1})/2}{l_{m-1}+2l_m+l_{m+1}}\Theta, \qquad \psi_{m-1} = l_m\Theta-\psi_m$$

$$2\theta_{m-1} = l_{m-1}\Theta-(\psi-\psi_{m-1}), \qquad 2\theta_{m+1} = l_{m+1}\Theta-(\psi-\psi_m), \qquad \psi = \tfrac{1}{2}h\Theta \quad (32a\text{–}e)$$

$$\psi_m \le \psi, \qquad \psi_{m-1} \le \psi. \quad (33)$$

The two conditions (33) are necessary for the two vertical members to satisfy the Foulkes condition and may be shown to be reduced to (31a,b), respectively. The conditions of non-negative hinge rotations: $\theta_{m-1} \ge 0$ and $\theta_{m+1} \ge 0$ are on the other hand reduced to (31c,d), respectively. It may therefore be concluded that the existence of the index m satisfying the inequalities (31) for a frame is the necessary and sufficient condition for the Design-IV to be the minimum weight design for the frame when $Q_m \le 0$.

It is apparent that the foregoing procedures of constructing the minimum weight designs may be extended and continued to those frames which have not been covered here.

5. EXAMPLES

Consider a class of frames with equal subspan length l and with $h = 1{\cdot}5l$. The shape function $L(m)$ may readily be obtained as follows:

$$L(m) = (l+h)l \quad \text{or} \quad -(l+h)l \qquad \text{if } f \text{ is even, and}$$

$$L(m) = 0 \qquad \text{if } f \text{ is odd.}$$

Figure 12 shows the regions of applicability of the four designs for the class of frames with $l_{m-1} = l_m = l_{m+1}$. For a frame with an even number of equal subspans, the line $\omega = (l+h)/2$ lies entirely within the region of Design-III as far as $0 \le h \le 2l$. For a frame with an odd number of equal subspans, the line $\omega = 0$ lies within the regions of Design-I, -II or -IV depending whether $h \le l$, $l \le h \le 3l/2$ or $3l/2 \le h \le 5l/2$, respectively. The two designs for $h = 1{\cdot}5l$ are indicated by the two circles in Fig. 12.

Figure 13 illustrates the two moment distributions at collapse for the two minimum weight designs of a Vierendeel frame, one for simply supported ends and the other for clamped ends. In observing the difference between the two designs, it should be noted that

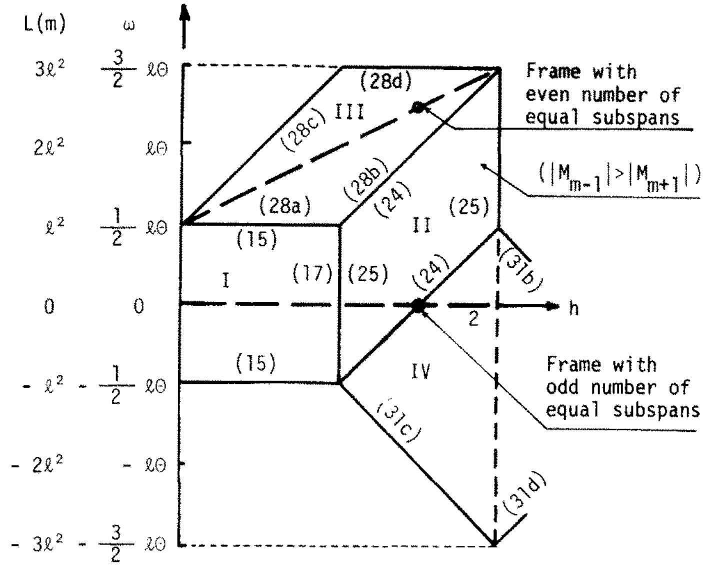

FIG. 12. Regions of applicability of the four designs for frames with $l_{m-1} = l_m = l_{m+1}$.

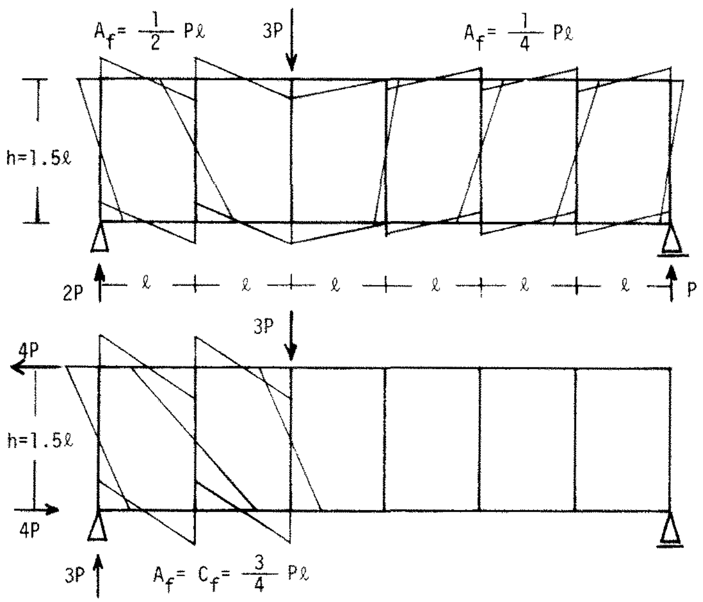

FIG. 13. The minimum weight designs and the corresponding moment diagrams at collapse of simply supported and clamped Vierendeel frames with equal subspan length.

the cost of exerting the clamping forces and the effect of axial forces have not been taken into account here.

6. EFFECT OF AXIAL FORCES

More realistic designs may be obtained by modifying for the effect of axial forces the minimum weight designs which have been obtained in the preceding sections. The fully plastic moment M^P for a cross-section must be such that the axial force N and the bending moment M acting upon the section do not violate the interaction yield condition. For example, the yield condition for an idealized sandwich section may be represented by

$$\frac{M^{PN}}{M^P} \pm \frac{N}{N^P} \pm 1 = 0 \tag{34}$$

where M^{PN} and N^P denote the fully plastic moment under the presence of N and the fully plastic axial force under the absence of M, respectively. The plastic moments A_i, B_i and C_i obtained in the preceding sections are M^{PN} in equation (34). As soon as N_i associated with M_i^{PN} of a sandwich member is determined, M^P defining the section may be obtained from

$$M_i^P = M_i^{PN} + \frac{M_i^P}{N_i^P}|N_i| = M_i^{PN} + \tfrac{1}{2}H_i|N_i| \tag{35}$$

for a choice of the depth H_i of the sandwich section. The axial force distribution corresponding to the moment diagram at collapse of a simply supported frame may immediately be calculated by statics and therefore the modified design may be obtained uniquely [9].

For a clamped frame, however, the axial force distribution depends upon the remaining redundant reaction, i.e. the overall fixed end moment when the frame is regarded as a clamped-clamped beam as a whole. Consider as an example a frame in Design-I. One of the axial forces, say N_m of the mth lower chord member (m being the minimizing index in Design-I) may as well be taken as the redundant force. Then in view of the moment diagram shown in Fig. 8(a), the axial force in the lower chord members at collapse may be expressed in terms of N_m as follows:

$$\begin{aligned} N_j &= N_m - \frac{2}{h}\sum_{i=j}^{m-1}(M_i + M_{i+1}) \qquad 1 \le j \le m-1 \\ N_k &= N_m + \frac{2}{h}\sum_{i=m+1}^{k}(M_{i-1} + M_i) \qquad m+1 \le k \le f \end{aligned} \tag{36}$$

The axial force in an upper chord member takes on the same magnitude as the corresponding lower chord member with the opposite sign. The axial force in the ith vertical member is simply given by $\frac{1}{2}P_i$. If the weight of the frame may again be assumed to be proportional to M^P, then N_m may be so determined, for a prescribed set of H_i's, as to minimize

$$g\sum M_i^P l_i = g\{\sum M_i^{PN} l_i + \tfrac{1}{2}\sum H_i|N_i|l_i\} = G + G_{NC} + G_{NV} \tag{37}$$

where

$$G = \sum M_i^{PN} l_i, \qquad G_{NV} = \frac{1}{2}h\sum_{s=0}^{f} P_s$$

$$G_{NC} = \sum_{j=1}^{m-1} H_j l_j |N_j| + H_m l_m |N_m| + \sum_{k=m+1}^{f} H_k l_k |N_k| \tag{38}$$

Since the minimum of G has already been achieved separately, it is reasonable to minimize G_{NC} alone with respect to N_m under the known set of frame moments, although the minimizations in the two separate steps may not necessarily provide the minimum of the modified weight as a whole. The minimum of G_{NC} may however be expected to provide a good upper bound on the minimum of the modified weight since the state of stress resultants so determined will satisfy all the equations of equilibrium, will determine a modified design just satisfying all the interaction yield conditions and hence may correspond again to an extremely deteriorated overcomplete collapse mechanism consisting of those plastic hinges which are governed by the plastic potential flow law associated with (34). It should be noted that the axial force diagram for chord members is of a staircase shape and coincide, when multiplied by h, with the overall moment diagram of the clamped-clamped beam only at the midspans of subspans. While Megarefs and Sidhu [10] are concerned with the minimum volume design of beams and frames with continuously varying cross-section, the present problem of modification is to minimize the absolute area of the axial force diagram of staircase shape weighted by H_r's with respect to one of the two redundant forces, or in other words, with respect to an up- or downward translation of the diagram. This is therefore quite similar to the previous problem of minimizing the absolute area of a subspan-shear diagram for G and the technique used in Section 4 may be applied, with the precaution that N_j and N_k are both monotonically nonincreasing from m toward 1 and f, respectively.

7. CONCLUSION

A general analytical method of constructing minimum weight designs has been proposed for Vierendeel frames. It has been shown that the new concept of the "frame moment" which has been introduced by Tanabashi and Nakamura [8] is useful and effective for understanding the overall beam character of a Vierendeel frame and for constructing its minimum weight design. It should be noted that the linear minimum weight design of an externally determinate Vierendeel frame is statically determinate without regard to its internal indeterminacy. The weight of an externally indeterminate frame has been minimized with respect to the external redundant forces. The four classes of designs obtained in simple and explicit analytical forms have sufficiently revealed the general intrinsic features of the linear minimum weight designs of clamped Vierendeel frames. The necessary and sufficient conditions for the minimum weight have been given in terms of the "shape function" introduced in the present paper. The minimum weight design of a clamped Vierendeel frame is almost generally such that each clamped end is to carry the vertical loads on those joints included in its respective half-span. It appears that this feature is essentially due to the circumstance that the cost of external supports has not been compared.

The present method is general enough to be extended to those clamped Vierendeel frames which have not been covered in this paper. The present approach and technique may also be applied in a fairly straightforward manner to continuous Vierendeel frames and other regular rectangular frames.

REFERENCES

[1] J. FOULKES, The minimum weight design of structural frames. *Proc. R. Soc.* **A223**, 482 (1954).
[2] R. K. LIVESLEY, The automatic design of structural frames. *Q. J. Mech. appl. Math.* **9**, 257 (1956).
[3] J. HEYMAN and W. PRAGER, Automatic minimum weight design of steel frames. *J. Franklin Inst.* **266**, 339 (1958).
[4] R. L. STONE, Automatic minimum weight design of steel frames on the IBM650 Computer. Brown Univ. Tech. Rep., IBM2038/2 (1958).
[5] J. J. KALKER, Automatic minimum weight design of steel frames on the IBM704 Computer. Brown Univ. Tech. Rep., IBM2038/3 (1958).
[6] H. TANAKA, Automatic analysis and design of plastic frames. *Rep. Inst. Ind. Sci. Univ. Tokyo* **12**, No. 3 (1962).

[7] T. KAWAI, Plastic analysis and minimum weight design of multi-story frames. 1965 Summer Conf. on Plastic Design of Multi-story Frames, Lehigh University, Bethlehem, Pa.

[8] RYO TANABASHI and TSUNEYOSHI NAKAMURA, The minimum weight design of a class of tall multi-story frames subjected to large lateral forces. *Trans. Architectural Inst. Japan* **118**, 10 (1965) and **119**, 37 (1966), and also *Proc. 15th Japan natn. Congr. appl. Mech.* p. 72 (1965).

[9] RYO TANABASHI and TSUNEYOSHI NAKAMURA, An approach to the last hinge point design of tall multi-story frames. *Proc. Symposium on External Forces and Structural Design of High-rise and Long-span Structures*, Japan Society for the Promotion of Sciences, p. 169 (1965).

[10] G. J. MEGAREFS and H. S. SIDHU, Simplifications in minimal design of frames. *J. struct. Div. Am. Soc. civ. Engrs.* **94**, 2985 (1968).

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Абстракт—В настоящей работе предлагается общий аналитический метод линейного расчета на минимум веса рам Виренделя, состоящих из призматических злементов постоянного поперечного сечения. Новая идея "рамного момента", предложенная Танабаши и Накамурой к расчету на минимум веса высотных многозтажных и многопролетных рамых зданий, оказывается полезной и зффективной, также для рам Вирендела. Предлягается расчеты на минимум веса для свободно опертых и защемленных рам Виренделя в простом и конелном ьцуе. Главным достоинством аналитических решений оказывается то, что общие характерные способы расчета на минимум веса могут быть представлены в простой форме. Дается, также, мотод расчета при учёте зффекта осевых сил.